Chapter 5.

The fundamental theorem of calculus.

In everyday language the word integration means the bringing together, or combining, of the parts, to make a unified whole. We might talk of:

a group integrating well into society,

the integration of the year sevens into high school,

the integration of three companies into one unified company,

a group of animals of a particular species being released from captivity and integrating with other animals, of the same species, already in the area,

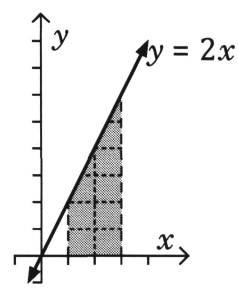
etc.

Hence the use of the word integration in mathematics when we are summing strips of area to make a unified whole. However, so useful is the fact that this *limit of a sum*, that we call integration, can be determined by antidifferentiating, that when asked to determine a definite integral we usually simply antidifferentiate automatically without thinking of the area context at all. For example, when question 3 part (a) of the previous Miscellaneous Exercise asked you to evaluate

$$\int_1^3 2x\,dx$$

it is likely that you proceeded algebraically, as shown below left, and not by considering areas, as shown below right.

$$\begin{aligned}\int_1^3 2x\,dx &= \left[x^2\right]_1^3 \\ &= 3^2 - 1^2 \\ &= 8\end{aligned}$$



Thus, asked to determine $\int_1^2 (3x^2 + 4)\,dx$ we do not need to consider limiting sums but can instead simply use our ability to antidifferentiate $3x^2 + 4$:

$$\begin{aligned}\int_1^2 (3x^2 + 4)\,dx &= \left[x^3 + 4x\right]_1^2 \\ &= \left((2)^3 + 4(2)\right) - \left((1)^3 + 4(1)\right) \\ &= 16 - 5 \\ &= 11\end{aligned}$$

Note • Remember that we do not need to include the "+ c" in evaluating a *definite* integral because, were we to include it, it would eventually cancel itself out.

However, if we are asked to determine $\int (3x^2 + 4)\,dx$, we now have an *indefinite* integral and the + c would need to be included.

Thus whilst
$$\int_1^2 (3x^2 + 4)\,dx = \left[x^3 + 4x\right]_1^2 = 16 - 5 = 11,$$
$$\int (3x^2 + 4)\,dx = x^3 + 4x + c.$$

• If we write $\int_a^b f(x)\,dx$, the answer is dependent only on the function and the values of a and b. The letter x is a "*dummy*" variable. We could replace it with a different letter and still obtain the same answer.

For example, whilst we found $\int_1^2 (3x^2 + 4)\,dx = 11$ it is also the case that

$$\int_1^2 (3t^2 + 4)\,dt = 11 \qquad \int_1^2 (3u^2 + 4)\,du = 11 \qquad \int_1^2 (3p^2 + 4)\,dp = 11$$

Exercise 5A.

For questions 1 to 3 first answer each question without the assistance of your calculator then, if you wish, confirm your answers with a calculator.

1. (a) Determine $\int (12t^2 + 6t)\,dt$

   (b) Determine $\int_1^x (12t^2 + 6t)\,dt$

   (c) Determine $\frac{d}{dx}\left(\int_1^x (12t^2 + 6t)\,dt\right)$

2. (a) Determine $\int \left(1 - \frac{1}{t^2}\right)dt$

   (b) Determine $\int_3^x \left(1 - \frac{1}{t^2}\right)dt$

   (c) Determine $\frac{d}{dx}\left(\int_3^x \left(1 - \frac{1}{t^2}\right)dt\right)$

3. (a) Determine $\int 2t\,(t^2+3)^4\,dt$

(b) Determine $\int_{-2}^{x} 2t\,(t^2+3)^4\,dt$

(c) Determine $\frac{d}{dx}\left(\int_{-2}^{x} 2t\,(t^2+3)^4\,dt\right)$

4. Use your answers to questions 1, 2 and 3 to complete the following:

$$\frac{d}{dx}\left(\int_{a}^{x} f(t)\,dt\right) = ????$$

Use the result from question 4 to determine each of the following, then check your answer on your calculator if you wish.

5. $\frac{d}{dx}\left(\int_{1}^{x} 4t\,dt\right)$

6. $\frac{d}{dx}\left(\int_{0}^{x} 5t^2\,dt\right)$

7. $\frac{d}{dx}\left(\int_{5}^{x} 2t^3\,dt\right)$

8. $\frac{d}{dx}\left(\int_{0}^{x} \frac{2t}{5-t}\,dt\right)$

9. $\frac{d}{dx}\left(\int_{5}^{x} (t+3)^4\,dt\right)$

10. $\frac{d}{dx}\left(\int_{0}^{x} 16t\,(t^2+3)^4\,dt\right)$

The fundamental theorem of calculus.

The fact that we can determine definite integrals using antidifferentiation, rather than referring back to their definition as a limiting sum, is very useful, as we found when determining the area under a curve.

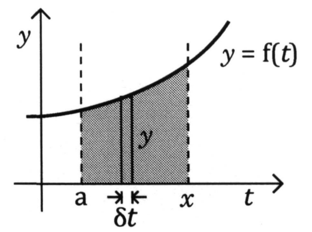
$$\int_{a}^{b} f(x)\,dx = F(b) - F(a)$$

where $F(x)$ is the antiderivative of $f(x)$.

Indeed so important is the fact that the limit of a sum, which we call a definite integral, can be determined using antidifferentiation it is referred to as **the fundamental theorem of calculus**.

Justification of the fact that the area under a curve, i.e. the limit of the sum of many rectangles, can be obtained using antidifferentiation, is given below.

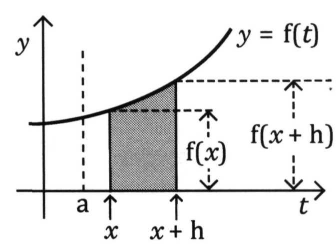
Consider the area under a curve from some fixed left hand boundary to a variable right hand boundary. The area, A, under the curve will then be a function of the variable right hand boundary. Suppose the curve is $y = \mathrm{f}(t)$, the left hand boundary is $t = \mathrm{a}$ and the right hand variable boundary is $t = x$.



$$\mathrm{A}(x) = \lim_{\delta t \to 0} \sum_{t=\mathrm{a}}^{t=x} y\,\delta t$$

$$\mathrm{A}(x) = \int_{\mathrm{a}}^{x} \mathrm{f}(t)\,\mathrm{d}t \qquad \leftarrow ①$$

For the diagram on the right:



The shaded area $= \mathrm{A}(x+h) - \mathrm{A}(x)$

But $h\,\mathrm{f}(x) < \mathrm{A}(x+h) - \mathrm{A}(x) < h\,\mathrm{f}(x+h)$

Thus $\mathrm{f}(x) < \dfrac{\mathrm{A}(x+h) - \mathrm{A}(x)}{h} < \mathrm{f}(x+h)$

I.e., the expression $\dfrac{\mathrm{A}(x+h) - \mathrm{A}(x)}{h}$ lies between $\mathrm{f}(x)$ and $\mathrm{f}(x+h)$.

Thus, as $h \to 0$ $\quad \dfrac{\mathrm{A}(x+h) - \mathrm{A}(x)}{h} \to \mathrm{f}(x)$, $\quad$ i.e., $\displaystyle\lim_{h \to 0} \dfrac{\mathrm{A}(x+h) - \mathrm{A}(x)}{h} = \mathrm{f}(x)$

Therefore $\quad \mathrm{A}'(x) = \mathrm{f}(x) \qquad \leftarrow ②$

Hence $\quad \mathrm{A}(x) = \mathrm{F}(x) + c \quad$ where $\mathrm{F}(x)$ is an antiderivative of $\mathrm{f}(x)$.

But the area from $x = \mathrm{a}$ to $x = \mathrm{a}$ must be zero thus:

$$0 = \mathrm{F}(\mathrm{a}) + c \qquad \text{giving} \qquad c = -\mathrm{F}(\mathrm{a})$$

and so $\quad \mathrm{A}(x) = \mathrm{F}(x) - \mathrm{F}(\mathrm{a})$

∴ Area from a to b is $\quad \mathrm{A}(\mathrm{b}) = \mathrm{F}(\mathrm{b}) - \mathrm{F}(\mathrm{a})$.

Thus $\quad \boxed{\displaystyle\int_{\mathrm{a}}^{\mathrm{b}} \mathrm{f}(x)\,\mathrm{d}x = \mathrm{F}(\mathrm{b}) - \mathrm{F}(\mathrm{a})} \quad$ where $\mathrm{F}(x)$ is the antiderivative of $\mathrm{f}(x)$.

Alternatively the previous boxed result can be written:

$$\boxed{\int_a^b f'(x)\,dx = f(b) - f(a)}$$

Equation ① on the previous page stated that $A(x) = \int_a^x f(t)\,dt$.

Equation ② on the previous page stated that $A'(x) = f(x)$.

From these it follows that $\boxed{\dfrac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x)}$

(A statement you may already be familiar with from Exercise 5A question 4.)

Thus $\boxed{\int_a^b f'(x)\,dx = f(b) - f(a)}$ and $\boxed{\dfrac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x)}$

From the rule above left we see that integrating the derivative of a function "gives us the function back".

and from the rule above right, differentiating the integral of a function "gives us the function back".

Hence the limit of a sum, which we call a definite integral, can be evaluated using antidifferentiation because integration and differentiation are opposite processes. This is what the **fundamental theorem of calculus** is all about.

The two boxed results above show the opposite nature of this relationship between the definite integral and differentiation. They are the two parts of the **fundamental theorem of calculus**. You used the rule above right to determine the answers to questions 5 to 10 of Exercise 5A.

Whilst the definite integral is the limit of a sum the fact that this integration process is the opposite of differentiation means that we tend use the integration symbol,

$$\int$$

without any values for a and b, to mean "find the antiderivative of" or, simply, "integrate", as you are already accustomed to doing.

Example 1

Find $\frac{d}{dx}\left(\int_0^x \frac{1+t^2}{2}\,dt\right)$.

With differentiation and integration being opposite processes we do not actually have to perform the integration, substitute in the values and then differentiate.

The fundamental theorem states that $\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x)$

Therefore $\frac{d}{dx}\left(\int_0^x \frac{1+t^2}{2}\,dt\right) = \frac{1+x^2}{2}$

$$\frac{d}{dx}\left(\int_0^x \left(\frac{1+t^2}{2}\right) dt\right)$$

$$\frac{x^2+1}{2}$$

Example 2

Integrate $\frac{d}{dx}(x^3 + 5x - 1)$ with respect to x.

With integration and differentiation being opposite processes we do not actually have to perform the differentiation as we would then only integrate our answer, and thereby obtain the initial function back again. However we do have to remember that an indefinite integral will give us a "+ c".

Thus $\int\left(\frac{d}{dx}(x^3 + 5x - 1)\right) dx = x^3 + 5x + c$

$$\int\left(\frac{d}{dx}(x^3+5x-1)\right)dx$$

$$x^3+5x$$

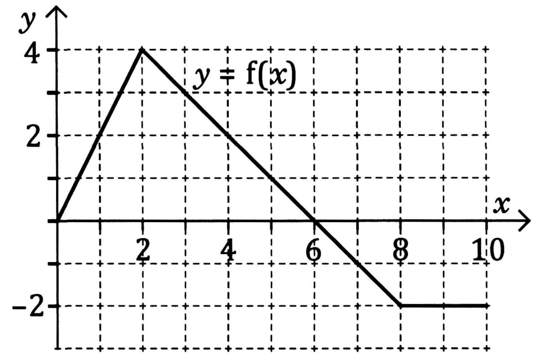
Example 3

For $0 \le x \le 10$ the function $y = f(x)$ is as defined by the graph on the right.

Determine:

(a) $f(2)$ (b) $f(7)$

(c) $\int_0^4 f(x)\, dx$ (d) $\int_2^{10} f(x)\, dx$

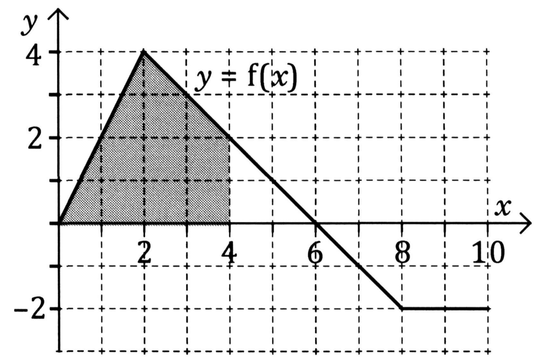


(a) When $x = 2$, $y = 4$. Thus $f(2) = 4$.

(b) When $x = 7$, $y = -1$. Thus $f(7) = -1$.

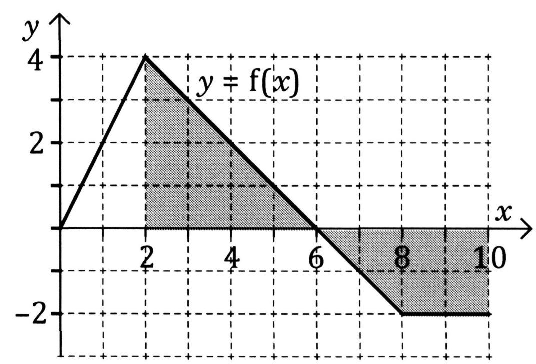
(c) Interpreting the definite integral as the signed area under a curve:

$$\int_0^4 f(x)\, dx = 10$$



(d) Similarly

$$\int_2^{10} f(x)\, dx = (8) + (-6)$$
$$= 2$$



Exercise 5B.

1. Find $\frac{d}{dx}\left(\int_0^x (2t + 3t^2)\, dt\right)$

2. Find $\frac{d}{dx}\left(\int_1^x (t^4 + 5)\, dt\right)$

3. Integrate the following with respect to x.

(a) $\frac{d}{dx}(x^2 + 5)$ (b) $\frac{d}{dx}(6x^3 - 4x^2 + 2x + 1)$

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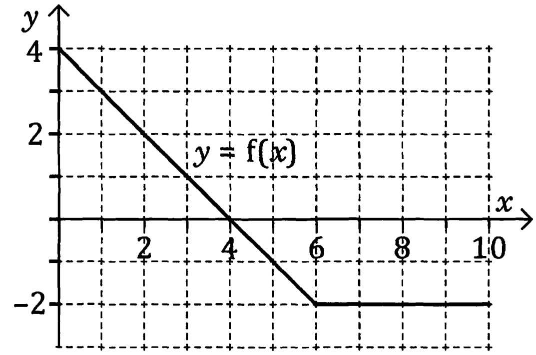
4. For $0 \le x \le 10$ the function $y = f(x)$ is as defined by the graph on the right. Determine:

   (a) $f(3)$ (b) $f(6)$

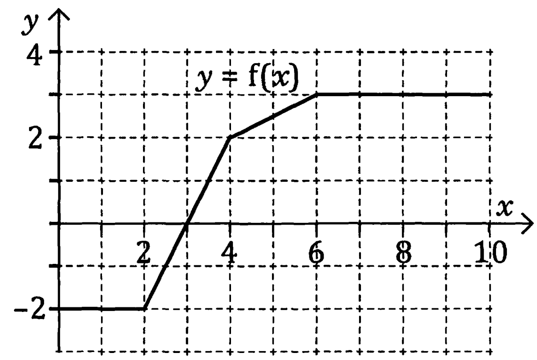
   (c) $\int_0^4 f(x)\,dx$ (d) $\int_0^{10} f(x)\,dx$



5. For $0 \le x \le 10$ the function $y = f(x)$ is as defined by the graph on the right. Determine:

   (a) $f(1)$ (b) $f(7)$

   (c) $\int_4^6 f(x)\,dx$ (d) $\int_1^8 f(x)\,dx$



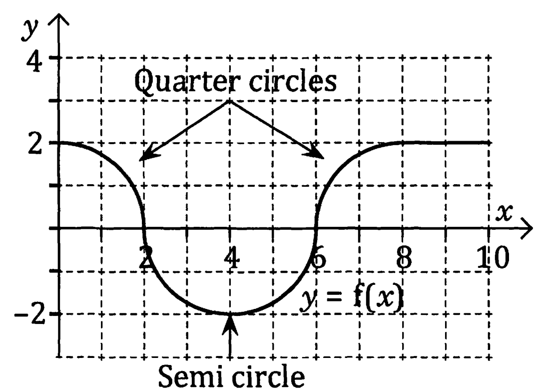
6. For $0 \le x \le 10$ the function $y = f(x)$ is as defined by the graph on the right. Determine:

   (a) $\int_0^2 f(x)\,dx$ (b) $\int_2^{10} f(x)\,dx$

   (c) For what values of a, $0 \le a \le 10$, is

   $$\int_0^a f(x)\,dx = 0$$

   (d) For what values of a, $0 \le a \le 10$, is $\int_0^a f(x)\,dx < 0$.



7. Find $\frac{dy}{dx}$ given that $y = x^2\sqrt{x} + \int_0^x (1 + 3t^2)^4\,dt$

Miscellaneous Exercise Five.

This miscellaneous exercise may include questions involving the work of this chapter, the work of any previous chapters, and the ideas mentioned in the preliminary work section at the beginning of the book.

For numbers 1 to 8 differentiate the given expression with respect to x.

1. $2x^3 + \sqrt{x}$
2. $(x + 5)(x - 3)$
3. $(3x - 1)^2$
4. $(3x - 1)^5$
5. $(5x - 1)(2x^3 - 3)$
6. $(5x - 1)(2x - 3)^3$
7. $\dfrac{2x + 3}{x - 1}$
8. $\dfrac{x - 1}{2x + 3}$

9. Clearly showing your use of the product rule, determine the gradient of the curve

$$y = (x - 1)\,(x^2 - 2)$$

at the point (0, 2).

10. (Without the assistance of a calculator.)
Use calculus techniques to determine the exact coordinates, and the nature, of any stationary points on the curve $y = x + \dfrac{6}{x}$.

11. (a) Describe what each of the responses 3, 28, 5 and 2 shown in the display on the right is informing us about the function $f(x)$ making sure that one of your statements includes the phrase

the average rate of change of $f(x)$

and another includes the phrase

the instantaneous rate of change of $f(x)$.

$f(0)$	
	3
$f(5)$	
	28
$\dfrac{f(5) - f(0)}{5}$	
	5
$\lim_{h \to 0}\left(\dfrac{f(x+h) - f(x)}{h}\right)\Big\vert\, x = 1$	
	2

(b) If the $f(x)$ that gave rise to the display on the right is a quadratic function,

i.e. $f(x) = ax^2 + bx + c,\ a \neq 0$,

find $f(x)$.

(c) If instead the $f(x)$ that gave rise to the display shown is a cubic function,

i.e. $f(x) = ax^3 + bx^2 + cx + d,\ a \neq 0$,

find a possible $f(x)$.

12. Find y in terms of x given that $\frac{dy}{dx} = 50(2x + 1)^4$ and $y = 7$ when $x = 0$.

13. Find $f(x)$ given that $f''(x) = 144(2x - 1)^2$, $f'(1) = 26$ and $f(1) = 6$.

14. (Without the use of your calculator.)
Find an expression for the marginal revenue if the total revenue, \$R, from the sale of x items is given by:

$$R = 30x - 0{\cdot}02x^2.$$

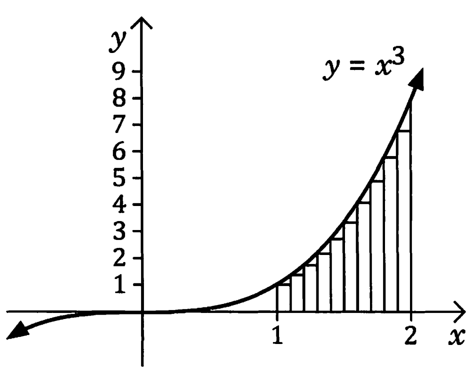
Find the marginal revenue when $x = 100$.

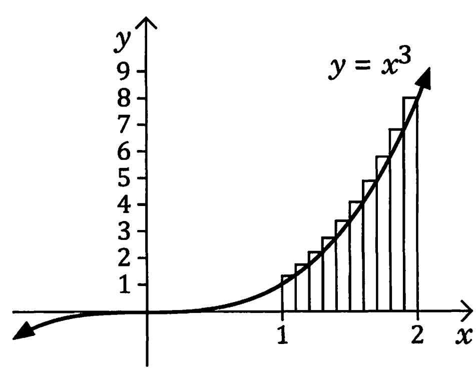
By approximately how much will the total revenue increase due to the sale of the 101st item?

15. Without the assistance of your calculator:
    (a) Produce a sketch of $y = (x + 1)(x - 2)^2$ showing the coordinates of any places where the curve cuts or touches the axes and of all stationary points and points of inflection.
    (b) Find the area enclosed by the x-axis and the curve $y = (x + 1)(x - 2)^2$.

16. Using ten rectangles in each case, as shown below, find an underestimate (using inscribed rectangles) and an overestimate (using circumscribed rectangles) for the area under

$$y = x^3 \text{ from } x = 1 \text{ to } x = 2.$$





and determine the mean of these two estimates.

17. The radius of a sphere is measured as 5 cm ± 0·1 cm. Use differentiation to find the volume of the sphere in the form $V\text{cm}^3 \pm b\text{cm}^3$ giving V and b to the nearest integer.